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EXHIBITING CHAOS AND FRACTALS WITH A MICROCOMPUTER

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Abstract—The microcomputer and its graphics capabilities are used to investigate chaos in Newton's method for a complex-valued quartic polynomial. Convergence maps that show fractal geometry are presented.

1. INTRODUCTION

Iterative methods have been used to find roots of equations for centuries. However, much is still unknown about the convergence properties or the structure of the domain of attraction and its complement for many of these methods. Microcomputer graphics can enhance the analysis and understanding of these properties and domains.

First, we provide a method for using computer graphics to investigate the domain of attraction. Then we discuss the basic principles of chaotic behavior and fractal geometry and describe our investigation of Newton's method for complex-valued polynomials. Finally, we provide several examples which show how these graphics can aid in the analysis. The examples clearly show that Newton's method can exhibit chaotic behavior, characterized by fractals.

2. GENERATION OF COMPUTER GRAPHICS

Computer graphics can provide a visualization of the geometry of the basins of attraction and the Julia set. Let the computer screen represent a region R of the complex plane. Assign to each root $\bar{z}$ of $f(z) = 0$ a color different from the colors assigned to any other root of $f(z) = 0$. Let $N \in Z^+$ and $\epsilon > 0$ be given (e.g. $N = 25$, $\epsilon = 10^{-3}$). Use each $Z \in R$ as an initial estimate for Newton's method and if $|Z_n - Z| < \epsilon$ for some $n \leqslant N$ assign z the color as $\bar{z}$ otherwise assign z a color different from the colors given to the roots. The result is an approximation to the convergence mapping for R originally proposed by Cayley [1].

3. CHAOS AND FRACTALS

Iterative numerical algorithms often exhibit chaotic behavior. Chaotic behavior occurs when a small change in a parameter completely changes the orderliness of the behavior in the outcome (cf. Hofstadter [2]). Chaos contrasts with stability which is characterized by small changes in a parameter resulting in small changes in the outcome. While this kind of unstable behavior has been known for a long time (cf. Julia [3]), study of such behavior has been rejuvenated in the last decade. This behavior is studied to discover patterns or conditions which explain the chaos. Saari and Urenko [4] used the theory of Julia [3] to analyze the random and periodic property of Newton's method. One discipline rich in chaotic behavior is numerical analysis, specifically iterative algorithms for nonlinear problems [5]. Another area with recent interest is biological systems [6].

A possible reason for this rejuvenation is the arrival of a new tool for the mathematician, the microcomputer (cf. Peterson [7]). Its computational and graphics capabilities provide visual imagery and hence possible new insights [8].

Chaotic behavior leads to structures which possess a fractal geometry. Fractals are geometric figures with the property that when a portion of the figure is magnified, that portion possesses the same geometric structure as the original figure (cf. Mandelbrot [9] and Peitgen and Richter [10]).

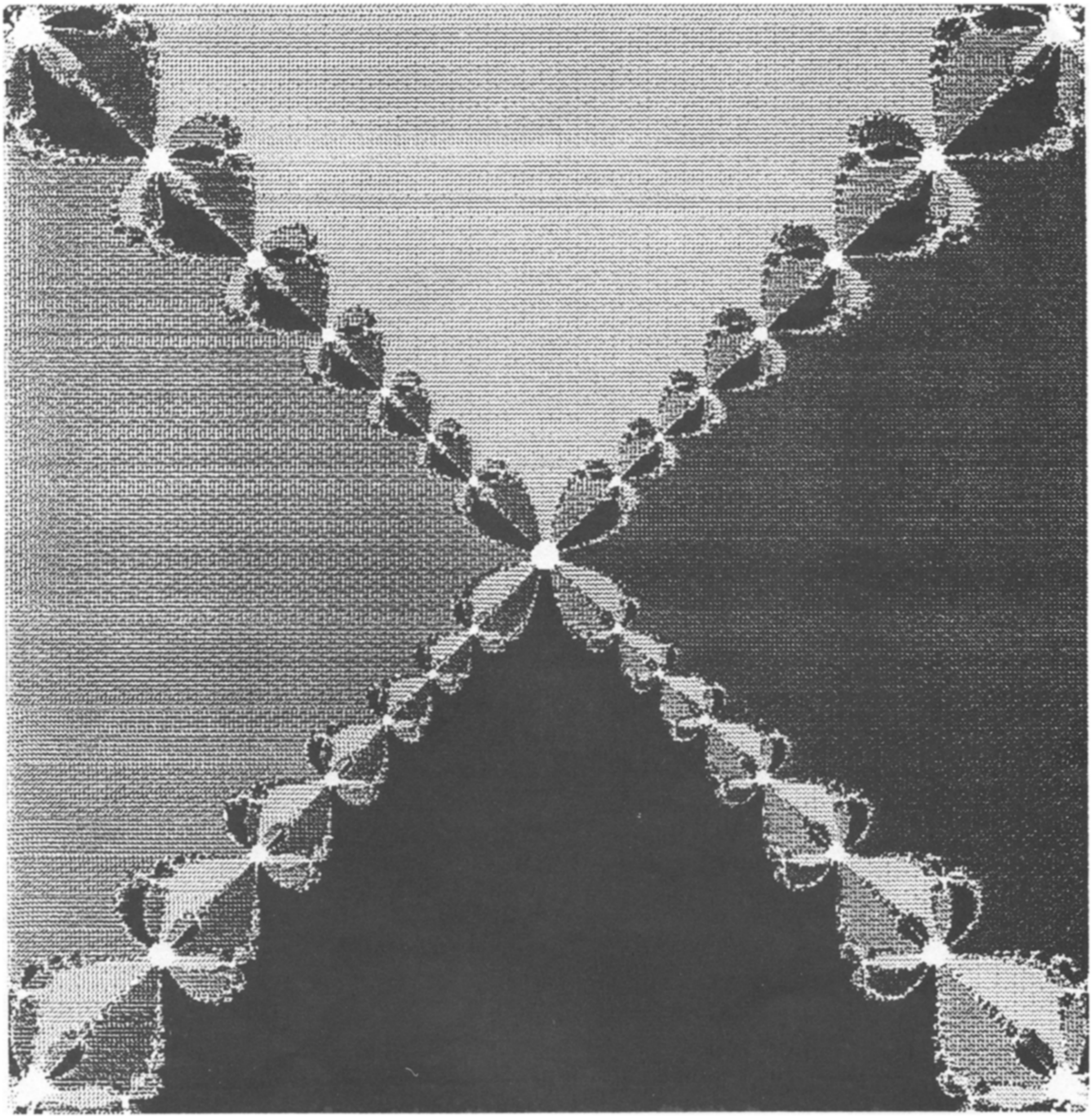

Fig. 1. Convergence map for Example 1, $\alpha = 1$. The large white regions are not nonconvergence zones of Newton's method. They are caused by programming measures we use to prevent computer overflows.

4. NEWTON'S METHOD

Let f be an analytic function of a complex variable $z = x + iy$. Newton's method can easily be used to approximate solutions of $f(z) = 0$. If z_0 is an initial approximation, then the $n + 1$th approximation is

$$z_{n+1} = g(z_n) = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{1}$$

If f is a polynomial of degree two or more, the root found by Newton's method usually depends on the location of z_0. However, the method fails to converge when z_0 falls on a boundary separating regions which converge to different roots. These regions of convergence are called *basins of*

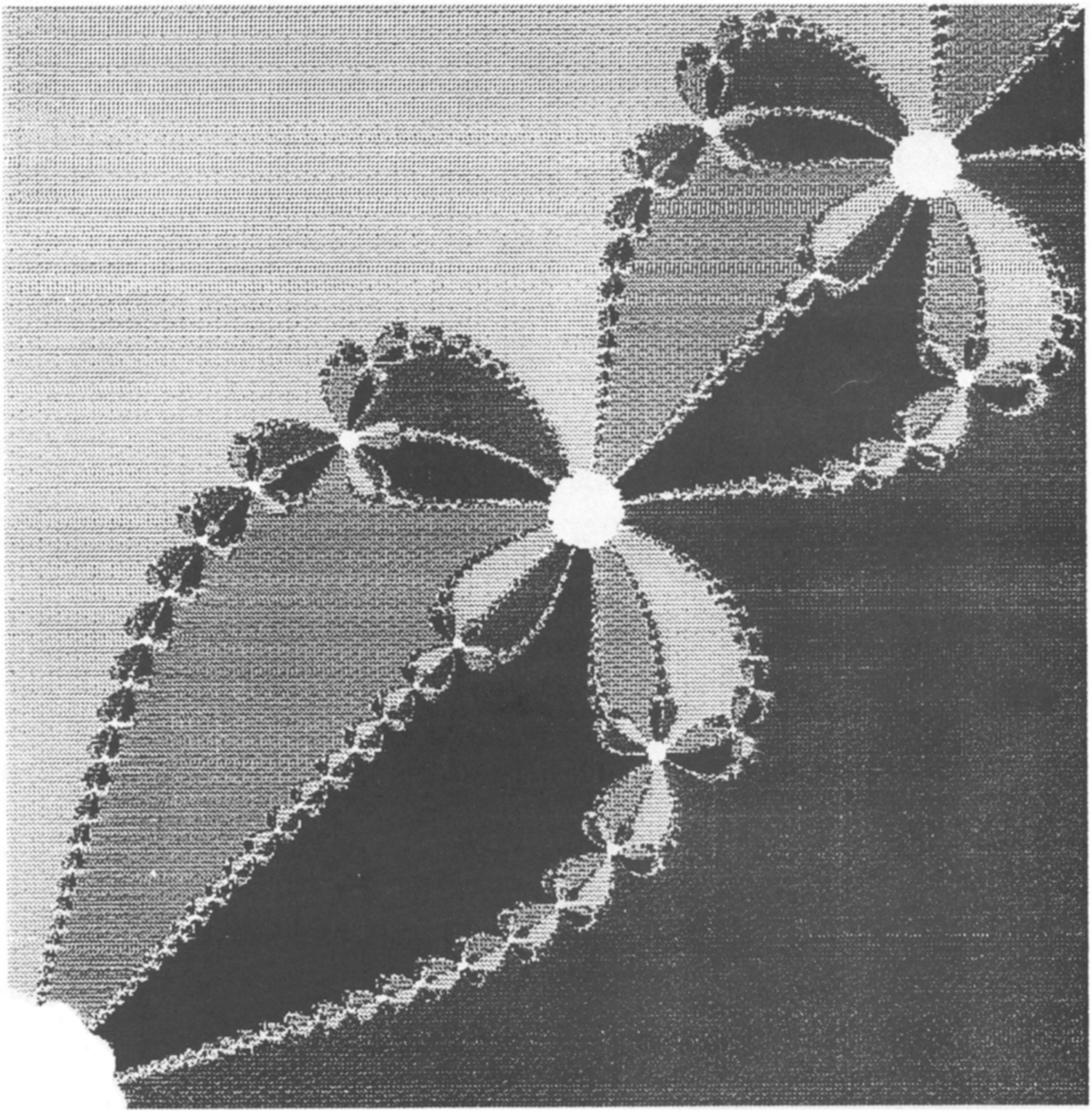

Fig. 2. Enlargement of the region ($0.0 \leqslant x \leqslant 1.0$, $0.0 \leqslant y \leqslant 1.0$) of the convergence map for Example 1, $\alpha = 1$.

attraction, and the roots are called *attractors*. The common boundaries of these basins of attraction constitute a Julia set. A Julia set is the closure of the set of all repelling periodic points of the iteration.

The repelling periodic points are those initial estimates z_0 which cause a cycle in the iteration $\{g(z_n)\}$, where for some $n = k$, $g(z_k) = g(z_0)$, and are repulsive since $g'(z_0) \cdot g'(z_1) \ldots \cdot g'(z_k) > 1$. Iteration of Newton's method for polynomials using initial points from the Julia set never converges to any root of the polynomial. Further explanation of repulsive cycles and the properties of Julia sets can be found in Ref. [11]. We call the set of starting points that cause the numerical method to fail to converge the nonconvergent set which, because of the numerical computation and computer roundoff, includes, but may not equal, the Julia set. Computational methods specifically designed to determine the Julia set for Newton's method are provided in Benzinger *et al.* [12].

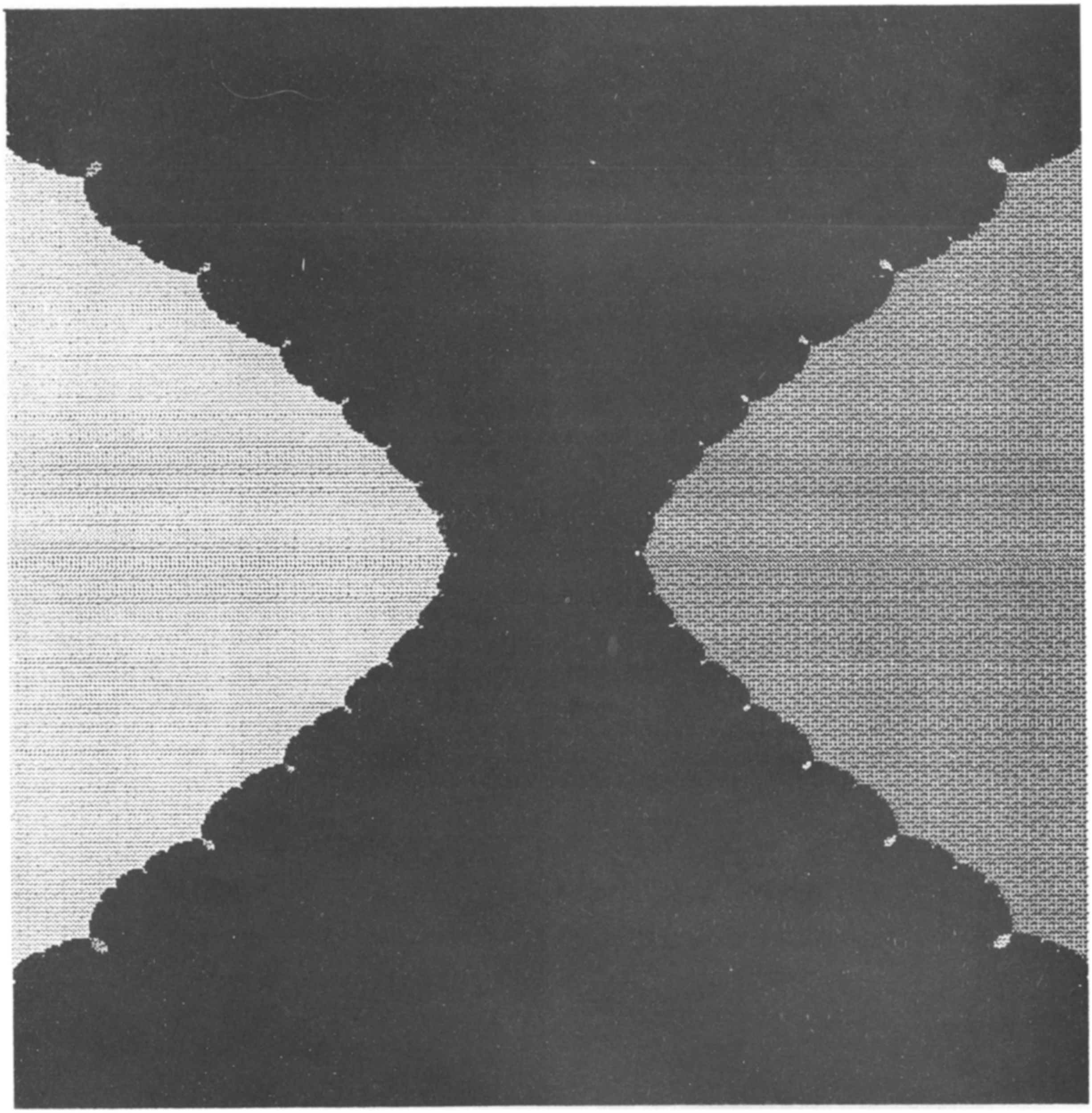

Fig. 3. Convergence map for Example 3, $\alpha = 0$.

5. EXAMPLES

Consider the complex-valued equation

$$f(z) = z^4 + (\alpha - 1)z^2 - \alpha = 0, \tag{2}$$

which has roots $z_1 = 1$, $z_2 = -1$, $z_3 = \sqrt{\alpha}i$, and $z_4 = -\sqrt{\alpha}i$. Different colors or shades are assigned to distinct roots. We used values of $N = 25$ and $\epsilon = 10^{-3}$ for the stopping criteria in the algorithm. Varying the value of the parameter α changes two of the roots. We chose to investigate this quartic with its one parameter because it allows for a single multiple root, two multiple roots, symmetric roots, and nonsymmetric roots. Therefore, by varying α, equation (2) can have characteristics of a quadratic, cubic or quartic by controlling the number of distinct roots to 2, 3 or 4, respectively.

We programmed the algorithm which produces the approximation to the convergent map in both Turbo-Pascal and BASIC on a Zenith Z-248 (IBM-AT compatible) microcomputer with an

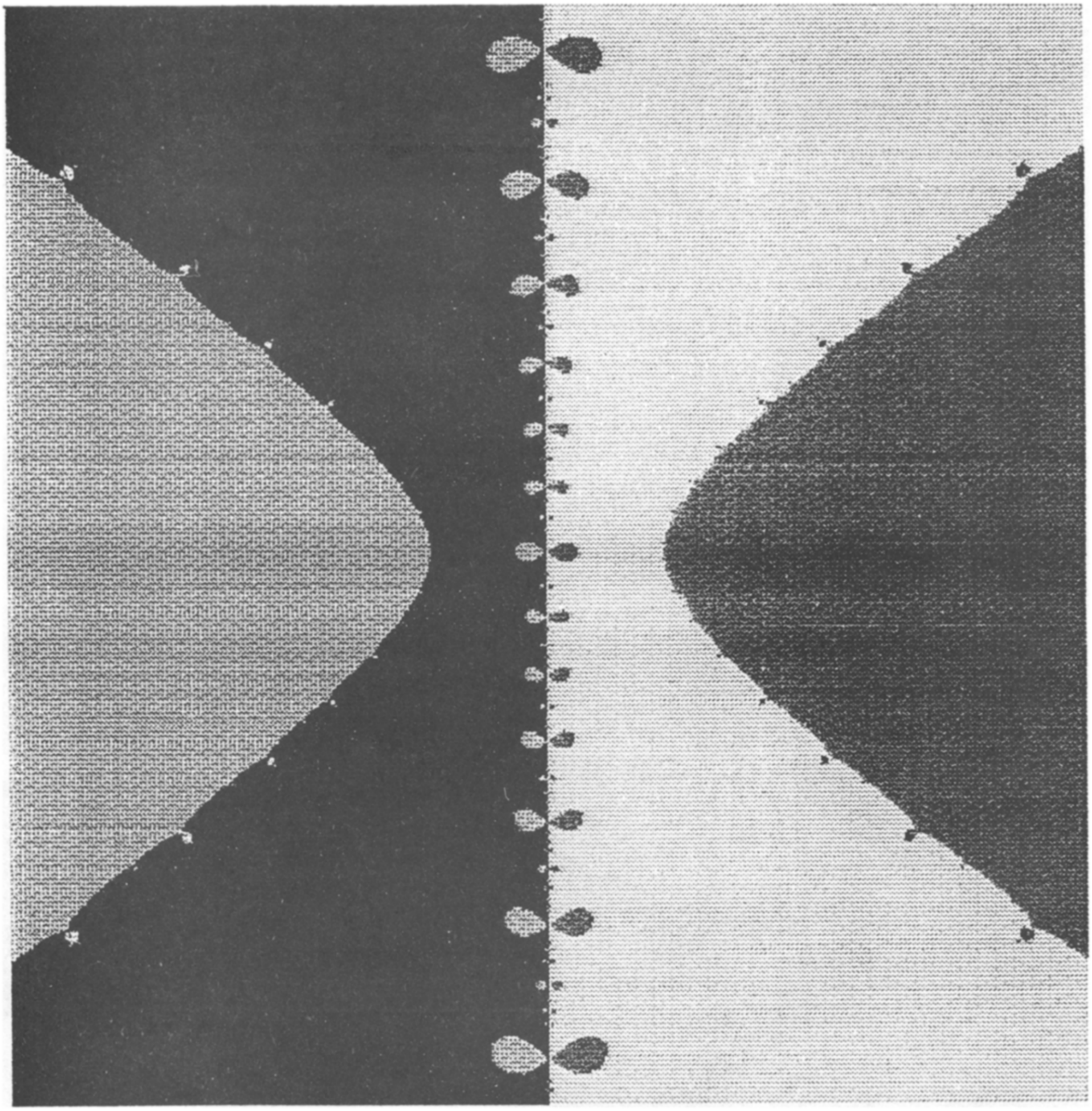

Fig. 4. Convergence map for Example 4, $\alpha = -\frac{1}{2}$.

enhanced graphics adaptor (EGA). We were able to use four colors and (320 × 200) pixel resolution in Turbo-Pascal, and 16 colors and (640 × 350) pixel resolution in BASIC to produce full-color screen displays on the microcomputer. The additional colors can be used to show rates of convergence. The figures shown in this paper are black and white versions with gray-intensity shading of these color images that were plotted on a Tektronix 4691 drum plotter. The nonconvergence points are shown in white.

Example 1

Setting $\alpha = 1$ in equation (2) produces the roots: 1, -1, i and $-i$. The convergence map for the domain of interest, $-4 \leqslant x \leqslant 4$ and $-4 \leqslant y \leqslant 4$, is provided in Fig. 1. The chaotic behavior in the regions near the lines $x = y$ and $x = -y$ is very apparent. The structures show a definite symmetry about the lines $x = y$ and $x = -y$ and are fractals since the patterns within

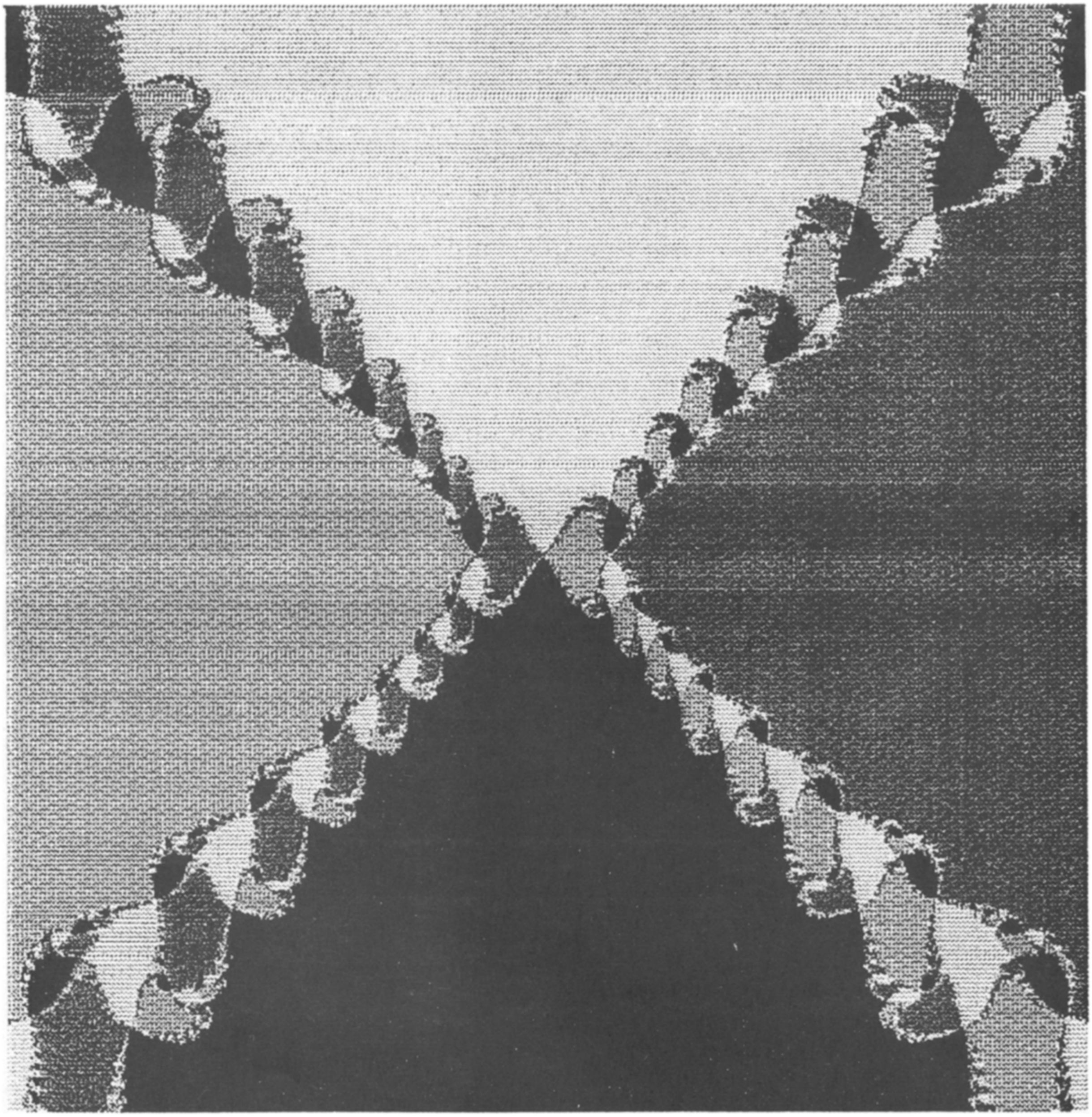

Fig. 5. Convergence map for Example 5, $\alpha = \frac{1}{2}$.

the structures are repeated at a finer and finer scale. This convergence pattern was not at all expected. Starting estimates can lead to convergence to roots on the opposite side of the domain, completely ignoring nearer roots. In the chaotic regions Newton's method is very sensitive to the starting estimate. Figure 2 shows the convergence map of the same problem with the domain of interest reduced to $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. This enlargement shows how the original patterns are repeated at finer and finer scales. Further enlargement continues to produce the same patterns.

Example 2

Setting $\alpha = -1$ in equation (2) produces the roots 1, 1, -1 and -1, which are two sets of repeated roots. This map shows no chaos. There are just two large basins of attraction. If z_0 is

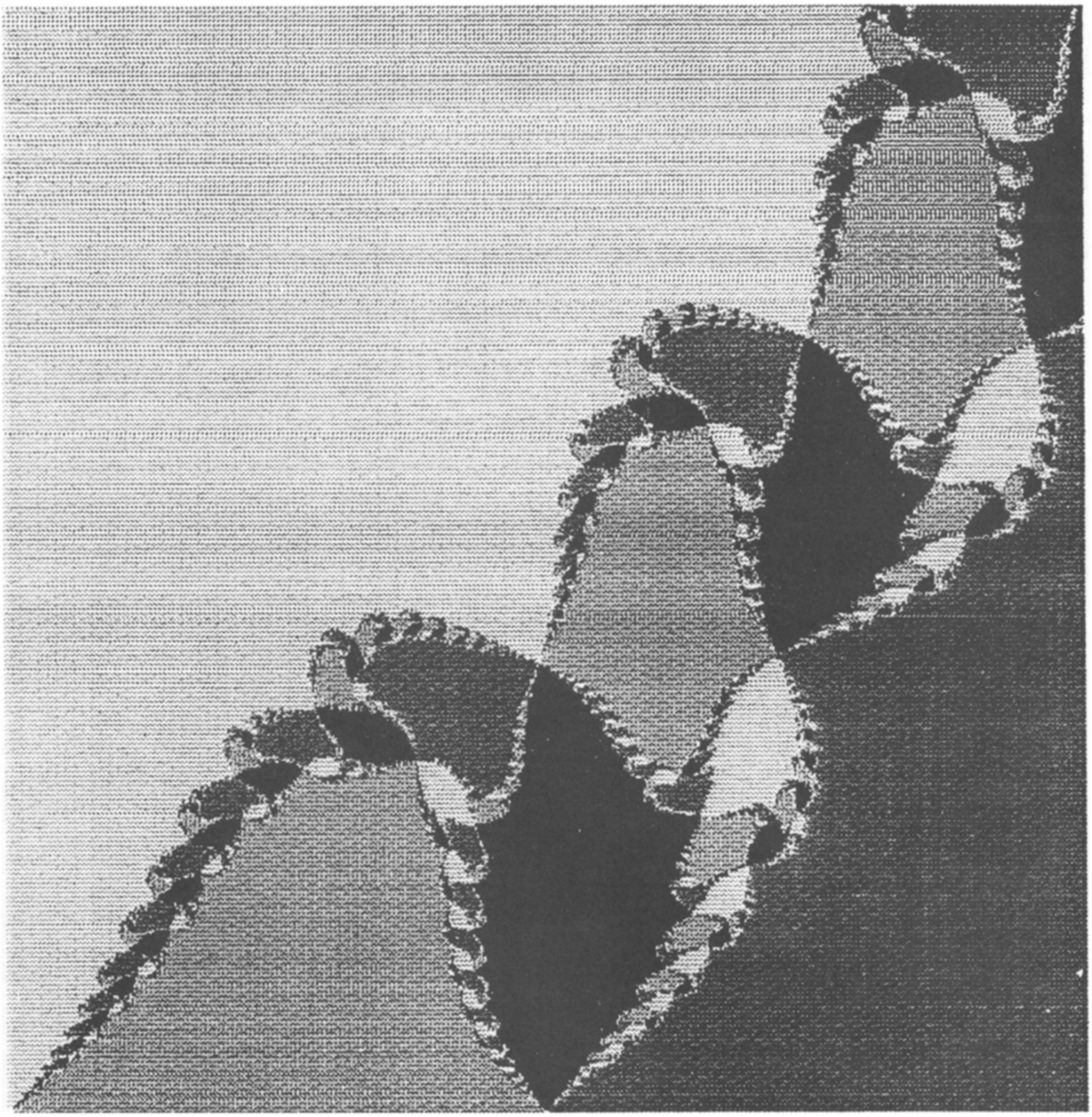

Fig. 6. Enlargement of the region ($0.0 \leqslant x \leqslant 1.0$, $0.0 \leqslant y \leqslant 1.0$) of the convergence map for Example 5, $\alpha = \frac{1}{2}$.

in the negative half-plane, its iterates converges to -1, while if in the positive half-plane, its iterates converge to 1. The imaginary axis is a nonconvergent boundary and includes the Julia set. This problem is similar to finding roots of the quadratic $z^2 - 1 = 0$. Root finding using Newton's method on complex-valued quadratics does not demonstrate any chaotic behavior.

The third-order cubic polynomial is the lowest-order complex-valued polynomial showing chaos using Newton's method [11]. Experiments with a generalized cubic were conducted by Benzinger *et al.* [12].

Example 3

Setting $\alpha = 0$ in equation (2) produces the roots -1, 0, 0, and 1, thus emulating a cubic. The convergence map for the domain of interest, $-4 \leqslant x \leqslant 4$ and $-4 \leqslant y \leqslant 4$, is provided in Fig. 3.

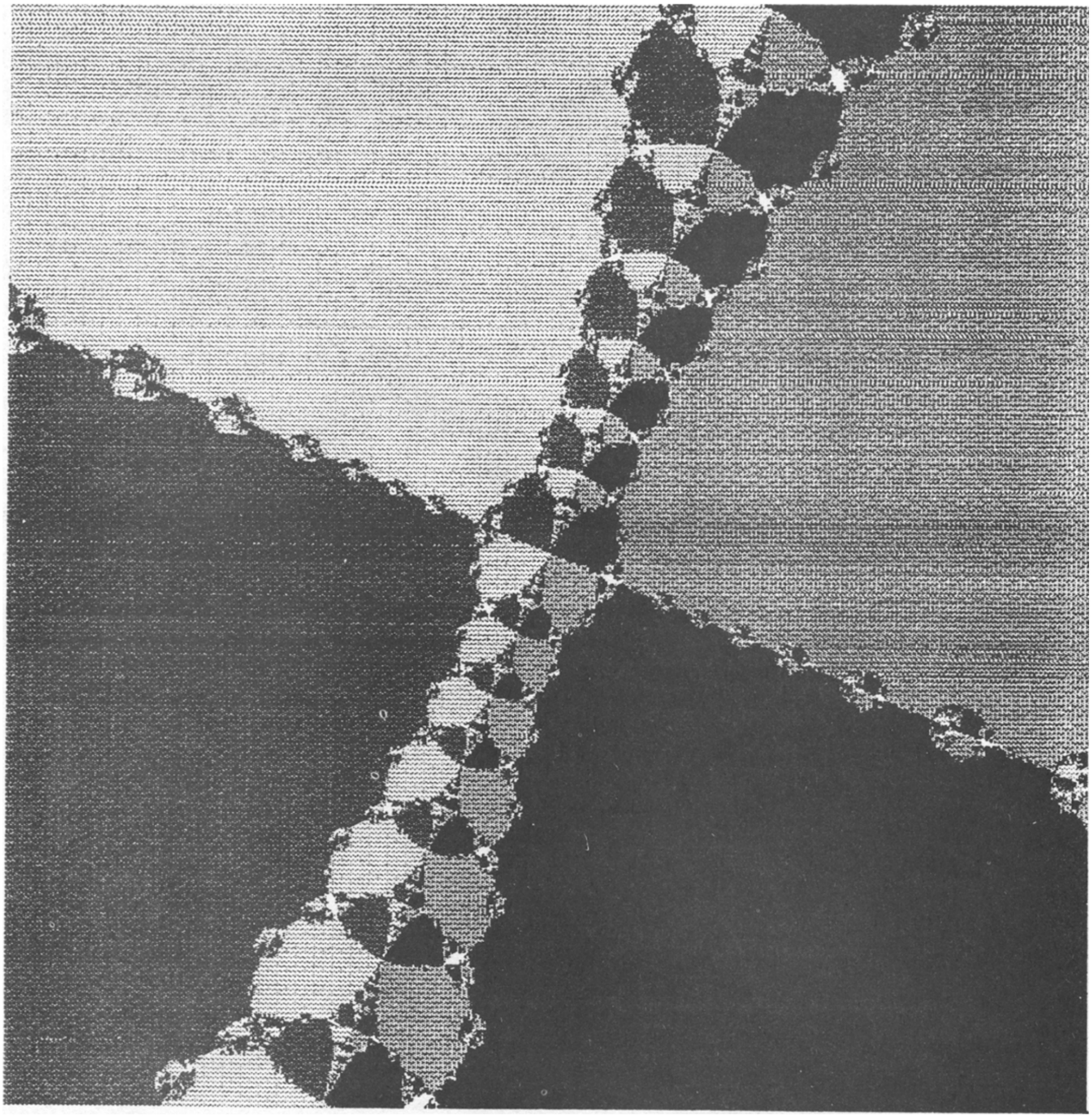

Fig. 7. Convergence map for Example 6, $\alpha = i$.

Again, there is chaotic behavior near the boundaries of the three basins of attraction. However, the patterns in the structures are simpler and less dramatic, they still maintain the nested patterns of a fractal.

Example 4

Setting $\alpha = \frac{1}{2}$ in equation (2) produces real distinct roots: 1, -1, $-1/\sqrt{2}$ and $1/\sqrt{2}$. The convergence map for the domain of interest, $-4 \leqslant x \leqslant 4$ and $-4 \leqslant y \leqslant 4$, is provided in Fig. 4. Once again, there is chaos near the boundaries of the basins of attraction with quite different types of structures and patterns existing in the different boundary regions.

Example 5

Setting $\alpha = \frac{1}{2}$ in equation (2) produces roots: 1, -1, $i/\sqrt{2}$ and $-i/\sqrt{2}$. The convergence map for the domain of interest, $-4 \leqslant x \leqslant 4$ and $-4 \leqslant y \leqslant 4$, is provided in Fig. 5. The four basins of

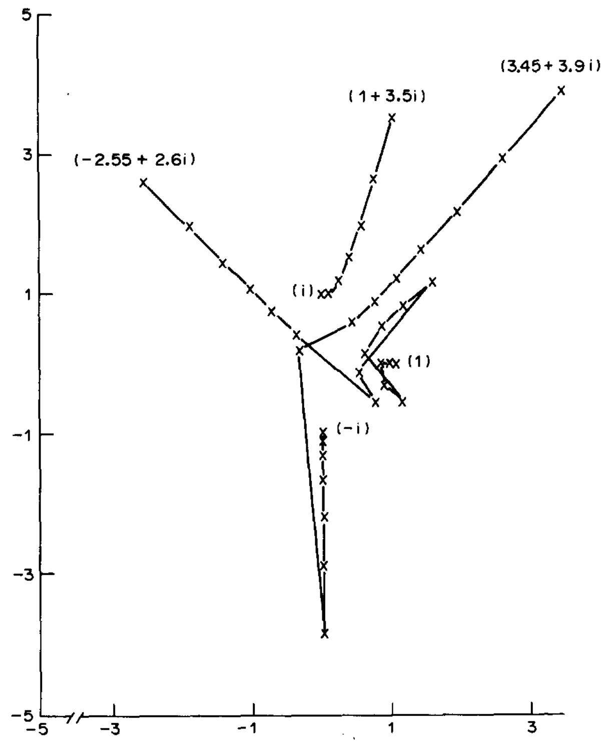

Fig. 8. Convergence trajectories for three initial estimates in Example 7.

attraction are similar to those in Example 1. However, the chaotic behavior is no longer symmetric with respect to the lines $x = y$ and $x = -y$. An enlargement of the region of interest, $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$, in Fig. 6 shows more of this intricate, chaotic structure.

Example 6

Setting $\alpha = i$ in equation (2) produces roots: 1, -1, $1/\sqrt{2} - i/\sqrt{2}$ and $-1/\sqrt{2} + i/\sqrt{2}$. The convergence map for the domain of interest, $-4 \leqslant x \leqslant 4$ and $-4 \leqslant y \leqslant 4$, is provided in Fig. 7. The difference in chaotic structures along the different boundaries of the basins of attraction is very apparent. The fractals in this example are quite different in structure than those in the previous examples, and there is no symmetry about the lines $x = y$ and $x = -y$.

Example 7

The sequence of values produced by Newton's method [equation (1)] for a given equation and given starting estimate can be traced using computer graphics. By drawing lines connecting the points in this sequence, trajectories of the convergence (or divergence) pattern can be drawn. This enables the trajectories for points in the chaotic regions to be analyzed in order to better understand the chaotic behavior and to search for patterns.

Figure 8 shows the convergence trajectories for three initial estimates for equation (2) with $\alpha = 1$. The trajectory (initial estimate $1 + 3.5i$) shows a direct convergence to the nearest root $z = i$. The other trajectories (initial estimates $3.45 + 3.9i$ and $-2.55 + 2.6i$) show strange convergence to $z = -i$, and $z = 1$, the most distant roots from the initial estimates. Figure 9 shows four more trajectories superimposed on a map of the nonconvergent set, which contains the Julia set. This map of the nonconvergent set is computed from the convergence map in Fig. 1 by changing colors of pixels of convergence points to white and changing color of pixels of nonconvergence points to black. All four of these initial estimates converge to the same root, $z = 1$, by marching toward the origin before being sent out into the basin of attraction. The symmetry of the initial estimates is maintained in their convergence trajectories.

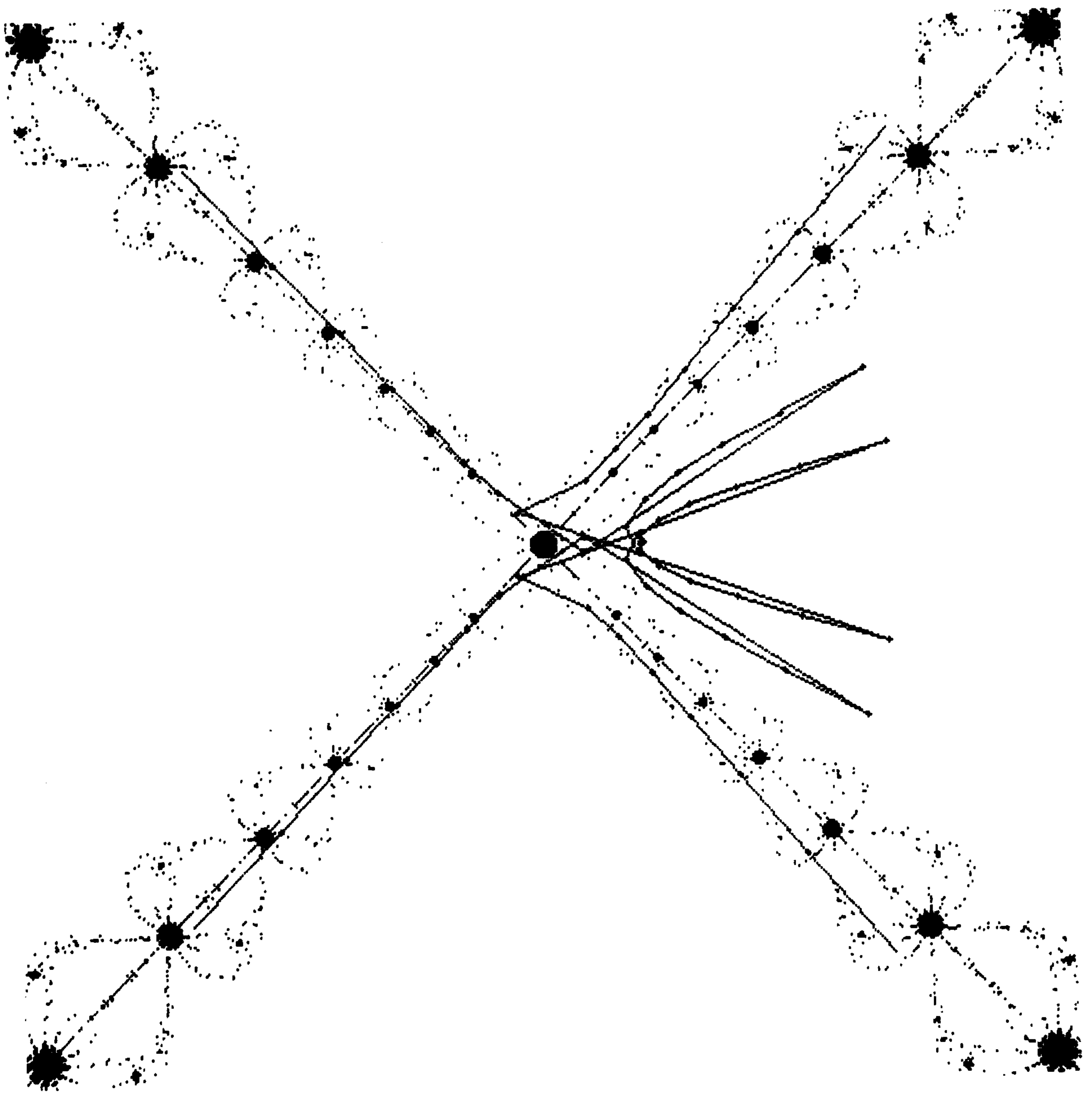

Fig. 9. Convergence trajectories for four initial estimates superimposed on the set of nonconvergence.

6. CONCLUSIONS

The graphical representation of the chaotic behavior of Newton's method is an important tool in demonstrating and understanding the mathematical phenomena of the method. The microcomputer with its graphics capabilities can provide the intuition to ask the questions and the verification to accept the answers in this investigation. Obviously, this work is not complete, and much more needs to be done to understand the chaotic behavior of Newton's method. We have also used the graphics capabilities of the microcomputer to animate the chaotic behavior as the parameter α in equation (2) varies. The animation provides further intuition of the dynamic and chaotic properties of this problem.

The capabilities of the microcomputer have been central in our investigation. It is the perfect tool to produce the mappings necessary to study the mathematics of chaos and fractals in this problem.

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REFERENCES

1. A. Cayley, The Newton–Fourier imaginary problem. *Am. J. Math.* **2,** 97 (1897).
2. D. R. Hofstadter, *Metamagical Themas*. Basic Books, New York (1985).
3. G. Julia, Mémoire sur l'itération des fonctions rationnelles. *J. Math. pure appl.* **1,** 47–245 (1918).
4. D. G. Saari and J. B. Urenko, Newton's method, circle maps, and chaotic motion. *Am. math. Mon.* **91,** 3–17 (1984).
5. A. R. Mitchell and J. C. Bruch, A numerical study of chaos in a reaction-diffusion equation. *Numer. Meth. partial diff. Equations* **1,** 13–23 (1985).
6. B. J. West and A. L. Goldberger, Physiology in fractal dimensions. *Am. Scient.* **75,** 354–356 (1987).
7. I. Peterson, Zeroing in on chaos. *Sci. News* **131,** 137–139 (1987).
8. C. A. Pickover, Mathematics and beauty: time-discrete phase planes associated with the cyclic system, $\{\dot{x}(t) = -f[y(t)], \dot{y}(t) = f[x(t)]\}$. *Computers Graphics* **11,** 217–226 (1987).
9. B. B. Mandelbrot, *The Fractal Geometry of Nature*. Freeman, San Francisco (1983).
10. H.-O. Peitgen and P. H. Richter, *The Beauty of Fractals*. Springer, Berlin (1986).
11. E. R. Vrscay, Julia sets and Mandelbrot-like sets associated with higher-order Schröder rational iteration functions: a computer study. *Math. Comput.* **46,** 151–169 (1986).
12. H. E. Benzinger, S. A. Burns and J. I. Palmore, Chaotic complex dynamics and Newton's method. *Phys. Lett.* **119A,** 441–446 (1987).